Hooray! Math!

Original credit for this handout goes to Eric Rosolowsky; modifications by Marshall Perrin.

Let's talk about MATH! You will need to know math for this class, but not too much, so it shouldn't scare you. We'll outline everything you need in the next few pages. Believe it or not, you almost certainly know all the math you will need for this class. Something happens between high school and college and I'm not exactly sure what it is. Still, I assure you (under oath) that you have seen this all before if you got into Berkeley. We just might disguise it a bit. Whatever you do, **DON'T PANIC**. This stuff giving you problems? Well, this might be your first opportunity to use your handy GSI! Feel free, otherwise we might just waste away from neglect.

- **Scientific Notation** – This notation is used throughout the core of astronomy because the numbers are just insanely huge and tiny. We all tend to lose sight of how these numbers relate to each other. Having a good grasp of this will help keep everything straight in your head.

  Scientific notation is just the trick of using exponents in conjunction with your favorite numerical base: 10. So, $10^3 = 10 \times 10 \times 10 = 1000$. No big whoop. Now, we know how to multiply two scientific notation numbers together: $10^2 \times 10^4 = (10 \times 10) \times (10 \times 10 \times 10 \times 10) = 1,000,000 = 10^6$. It's often said that the exponent indicates the number of zeros that follow the 1. This is great for positive exponent but gets confusing when we have to deal with **negative exponents**.

  Negative exponents on that 10 indicate division by powers of 10, not multiplication. So $10^{-2} = 1 \div (10 \times 10) = \frac{1}{100} = 0.01$. Again, multiplying numbers adds exponents: $10^{-3} \times 10^{-4} = 10^{-3+-4} = 10^{-7}$. In parallel with multiplication, division of numbers means that we subtract exponents (top minus bottom). So, $10^5 \div 10^2 = \frac{100{,}000}{100} = 1000 = 10^3 = 10^{5-2}$. This can get a little tricky with fraction bars indicating division:

  $$\frac{10^{-3}}{10^7} = 10^{-3-7} = 10^{-10} \qquad \text{or} \qquad \frac{10^{-3}}{10^{-7}} = 10^{-3-(-7)} = 10^{-3+7} = 10^4$$

  Yup, all those little details about negative signs and multiplication are back! Feeling patronized? Well, take that as a sign that the math is nothing you cannot handle!

- **Mathematics with Scientific Notation** – Not all numbers are pure powers of 10, as much as astronomers may behave that way. To represent these other numbers, we just multiply the appropriate powers of 10 by a number between 1.0 and 9.9999... One of our favorite numbers is the speed of light, represented by the variable c. In our standard units, $c \approx 2.998 \times 10^8$ meters per second. We'll learn all kinds of crazy stuff about this number, but right now, we'll just focus on the fact that $c = 2.998 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$ m/s. This is nothing too exciting, except for the fact we need to focus on how to multiply and divide with this kind of number. The trick is just to do the math with the powers of 10 and the other numbers separately, combining it at the end. So. $(3 \times 10^8) \times (2 \times 10^4) = (3 \times 2) \times (10^8 \times 10^4) = 6 \times 10^{12}$. Remember that the number in front needs to be between 1 and 10; thus $(7 \times 10^4) \times (6 \times 10^3) = 42 \times 10^7 = 4.2 \times 10 \times 10^7 = 4.2 \times 10^8$.

  Division works the same way:

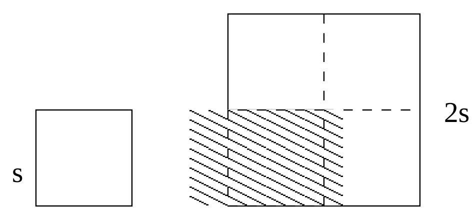
  $$\frac{5 \times 10^7}{4 \times 10^{-4}} = 1.25 \times 10^{7-(-4)} = 1.25 \times 10^{11} \qquad \frac{\frac{7}{5} \times 10^{-6}}{2 \times 10^{-4}} = \frac{7}{10} \times 10^{-6-(-4)} = 0.7 \times 10^{-2} = 7 \times 10^{-3}$$

- **Subscripts** – Astronomers just love subscripts on their variables. Subscripts are just another way to name variables, by hanging numbers or symbols off of them. If p is a variable then p_1 is a new variable with subscript 1. Remember, that subscripts are just ways of distinguishing variables so p_1 is not the same as p_2. This way of naming variables is convenient since a variable often represents a physical quantity, say d for distance. Thus d_1 and d_2 can mean the distance in two different situations, say to object 1 and object 2. Or M_{Mars} and $M_{Jupiter}$ are the masses of the next two planets in the solar system. Using the astronomical symbols for the sun ($\odot$) and the earth ($\oplus$), we could say that $d_\odot$ is the distance to the sun and $r_\oplus$ could be the radius of the earth. Just remember, the subscripts are part of the variable name, nothing more.

- **Proportionalities (a.k.a. Ratios** – This is the other big thing that astronomers love to bring into their beginning astronomy courses. The notation notation for this may be the only math in this course you haven't seen before, but the concepts are simple. Once you get ahold of what we're trying to communicate, it becomes a very powerful tool for the kind of mathematical reasoning that we'll be using here.

  The basic idea with proportionalities is that we want to talk about, *in general*, how two different quantities relate, *without worrying too much about the specifics.* For instance, I can tell you that the farther you are away from campus, the longer it will take you to walk to school, without having to say specifically where you are or how fast you walk.

  Here's another example: Think about a square field which has fence on its four sides. Now, think about what happens if we surround a field with fences that are twice as long as the ones that are around the first field. The question that your astronomy professor wants to ask is how much area is enclosed in the second field relative to the first. This question is addressed best by a picture: So, just by looking at the picture, we see that four times the area is enclosed by the second fence.



  $A_2 = 4 \times A_1$.

  So, let's turn to how an astronomer would express this problem. He/She would make the statement that the area contained is proportional to the square the length of the sides of the field. Or, using mathspeak: $A \propto s^2$. **Proportionalities are used in calculations when comparing two situations** (what a great opportunity to use subscripts). What the proportionality symbol ($\propto$) really means is that this is an equation with some number in the equation that we can ignore in the calculation. Why can we do this? Well, read on...

  In the case of the fields above the area (A) equals the length of the side squared (s^2). So, $A = s^2$ and the number making $A \propto s^2$ into an equation is just 1. So this is a pretty simple example. The main point of that example is that when you double the side length of the square, the area goes up by a factor of 4. Now let's try a slightly trickier one. So, let's consider the situation of a circle where the area enclosed $A = \pi r^2$. With proportionalities, we can write $A \propto r^2$ and the constant of proportionality is π. The expression $A \propto r^2$ lets us write down the general relation between A and r, without worrying about the little details like what π's value is and so on.

  You should be asking how we use all this wonderful information! Let's consider the following problem:*Circle 1 has a radius that's 5 times that of Circle 2. By what factor does Circle 1 exceed Circle 2's area?* Aieee! Algebra flashback! But no worries: Our method for solving this problem is to consider ratios. We want the ratio of the areas A_1/A_2 (note the nifty subscripts). We know the ratio of the radii: $r_1/r_2 = 5$. To solve the problem, we write down the problem in two situations:

$$A_1 = \pi r_1^2 \qquad A_2 = \pi r_2^2$$

Then, we divide the two equations to get a new equation:

$$\frac{A_1}{A_2} = \frac{\pi r_1^2}{\pi r_2^2} = \left(\frac{r_1}{r_2}\right)^2$$

Aha! We know $r_1/r_2 = 5$, so $A_1/A_2 = 5^2 = 25$. There's our answer! The most important thing about this method is that the π disappeared – it was in the top and bottom of the fraction and we quietly cancelled it. All ratio problems are typified by this kind of cancellation.

As a second example, consider the Stefan-Boltzmann law which says that the luminosity of an object L is proportional to r^2T^4 ($L \propto r^2T^4$). (You're not expected to recognize this formula! We're just using it here for an example of how the math works.) The actual equation reads $L = 4\pi\sigma r^2T^4$, but $4\pi\sigma$ is a constant of proportionality, and we want to work through the problem without having to calculate it. We'll get an answer without me telling you what σ actually is!

Problem: *The sun's radius is 100 times bigger than that of the earth. $T_\odot = 6000\ K$ and $T_\oplus = 300K$. How much more luminous is the sun than the earth?*

Let's work through the problem knowing the equation and writing it down in two situations:

$$L_\odot = 4\pi\sigma r_\odot^2 T_\odot^4 \qquad L_\oplus = 4\pi\sigma r_\oplus^2 T_\oplus^4$$

Dividing the two equations gives:

$$\frac{L_\odot}{L_\oplus} = \frac{4\pi\sigma}{4\pi\sigma}\left(\frac{r_\odot}{r_\oplus}\right)^2\left(\frac{T_\odot}{T_\oplus}\right)^4 = 100^2 \times 20^4 = 1.6 \times 10^9$$

Now, this is so repetitive that there has to be a trick! There is: **To make a proportionality into an equation, just write down ratios of variables in place of variables.** So, where we see $L \propto r^2T^4$, this becomes an equation replacing L with L_1/L_2, r with r_1/r_2 and T with T_1/T_2:

$$L \propto r^2T^4 \Longrightarrow \frac{L_1}{L_2} = \left(\frac{r_1}{r_2}\right)^2\left(\frac{T_1}{T_2}\right)^4$$

This is just what we had before without worrying about the constant of proportionality!

Here's another example we'll use a lot. The inverse square law of light says that $f \propto 1/d^2$ where f is the amount of light we receive (called the flux) and d is the distance to the object. So, if we view an object from two different distances (d_1 and d_2), then the ratio of fluxes is

$$\frac{f_1}{f_2} = \frac{1}{(d_1/d_2)^2} = \left(\frac{d_2}{d_1}\right)^2$$

Note that proportionality symbols behave a lot like equals signs. Here are two properties we'll use a lot:

$$a \propto b,\ b \propto c \Longrightarrow a \propto c$$

$$a \propto \frac{1}{b} \Longrightarrow \frac{1}{a} \propto b$$

- **Dimensional Analysis** – Though the name is cool, this part is actually about *units* on physical quantities. We need units badly because we have all kinds of different measurements to take. NASA needs units and when they don't do the dimensional analysis correctly, they smack millions of taxpayer dollars into the face of Mars. No foolin'. Our goal for this course is to never make the same mistakes that they did. There are two rules!

  1. Always write units on answers. Some answers (like ratios) don't have units.
  2. Convert between units by multiplying by a conversion factor equal to 1.

  You will *never* go wrong if you remember to use these two rules. The first is self explanatory. As Dr. Gaustad, my mentor in college told me, "If an answer doesn't have units, it's wrong." How far away is the sun? 1? 9.3×10^6? 1.5×10^{11}? All of these can be right, with the proper units, but by themselves they're all wrong!

  The second rule tells you how to change between different units. You start with an expression that relates two units, like

  $$1 \text{ A.U.} = 1.4 \times 10^{11} \text{ m}$$

  Then you divide through by one side or the other so you get a 1 on one side:

  $$1 = \frac{1 \text{ A.U.}}{1.4 \times 10^{11} \text{ m}} \qquad \text{or} \qquad 1 = \frac{1.4 \times 10^{11} \text{ m}}{1 \text{ A.U.}}$$

  Now we have two convenient factors to multiply expressions by without changing them (since multiplying by 1 doesn't change the value of the expression). In this case, we can use the fractions we have to convert any number from meters to A.U. or vice versa.

  Example:A typical problem might read: *Calculate the distance from the sun to Jupiter in kilometers if it is 4.5 A.U. away from the sun.*

  To start on the problem, we write down the first expression and multiply by whichever factor of 1 that we have that cancels the units we don't want and gives us the units we want.

  $$4.5 \text{ A.U.} = 4.5 \text{ A.U.} \frac{1.4 \times 10^{11} \text{ m}}{1 \text{ A.U.}} = 6.3 \times 10^{11} \text{ m} = 6.3 \times 10^{11} \text{ m} \frac{1 \text{ km}}{1000 \text{ m}} = 6.3 \times 10^{8} \text{ km}$$

  Note that in the last step, we have used the conversion factor 1 km = 1000 m. This gives us the conversion factor for the final step.

- **Prefixes** – We use tons of prefixes in front of units as short-hand for scientific notation. A full table of these is found on page A-1 of your book's appendix. To use this table, we find a unit like nW. W is the symbol for Watts, a unit of power, and n is the prefix for nano- as we see in the table. Nano-means 10^{-9} in scientific notation, so 1 nW = 10^{-9} W and we have a conversion factor we can use in dimensional analysis.

- **Scaling Problems** – This work on dimensional analysis is very important for another problem that we do a lot at the beginning of the course. Essentially, it's building a scale model. The typical problem might read: *Imagine shrinking the size of 1 parsec (an astronomical unit of distance, abbreviated* pc*) to 1 centimeter. How far is it to the center of our galaxy from our sun (8.5 kpc) in this model?*

  To solve this problem, we need an equation between units like we had for dimensional analysis. The problem tells us that 1 model pc = 1 real cm. We just use this conversion factor like we would in dimensional analysis and start our dimensional analysis chain:

  $$8.5 \text{ model kpc} = 8.5 \text{ model kpc} \frac{1000 \text{ model pc}}{1 \text{ model kpc}} = 8500 \text{ model pc} \frac{1 \text{ real cm}}{1 \text{ model pc}}$$

$$= 8500 \text{ real cm} \frac{0.01 \text{ real m}}{1 \text{ real cm}} = 85 \text{ real m}$$

Sample Problems

1. Express the following in scientific notation:

$$\text{a.)}34,000,000 \quad \text{b.)}0.000302 \quad \text{c.)} \quad 2 \times 10^5 \times 8 \times 10^{-9} \quad \text{d.)}\frac{9 \times 10^8}{3 \times 10^3} \quad \text{e.)}\frac{4 \times 10^{-4}}{5 \times 10^{-13}}$$

2. As mentioned previously, the inverse square law of light says that the flux of light we see is inversely proportional to the square of the distance between us and the light source ($f \propto 1/d^2 \propto d^{-2}$). This law holds for the sun. So, what is the ratio of the flux of sunlight between us on earth and martians on Mars (d=1.5 A.U.). How about Mercury (d=0.4 A.U.). Based on this, which planet has a hotter surface?

3. California consumes roughly 33 Gigajoules (33 GJ) of energy in one second.
   (a) Express this number in Joules, using scientific notation.
   (b) How many calories is this? (1 calorie = 4.2 J).
   (c) How many Mountain Dews is this? (1 Mtn. Dew contains 170 kcal).

4. For an example of scaling problems, see your homework this week. (And you say I teach you nothing useful...)