Ay 7A – Fall 2009
Section Worksheet 12
Black Holes and Revelations[1]

1. **Tidal Forces and Spaghettification**[2]

   (a) Prof. Chiang is so sad that Ay 7A is almost over that he dives out of his spaceship head first and floats toward a black hole. What is the gravitational force on his head (F_h) due to a black hole of mass M when his head is a distance r from the center of the black hole? Assume Prof. Chiang has a mass m.

$$F_h = \frac{GMm}{r^2}$$

   (b) Now let's write down an expression for the force on his feet (F_f), assuming that the professor's height is x.

$$F_f = \frac{GMm}{(r+x)^2}$$

   (c) Subtract these two values to get the *difference* in forces across Prof. Chiang's body. This is called the "tidal force" on the professor.

$$\Delta F = F_h - F_f = \frac{GMm}{r^2} - \frac{GMm}{(r+x)^2}$$

$$\Delta F = GMm\left(\frac{1}{r^2} - \frac{1}{(r+x)^2}\right)$$

   (d) Now assume that $r \gg x$ (i.e., the distance to the center of the black hole is much, much larger than the professor's height) and Taylor expand the above expression for the tidal force (as usual, anything $\ll 1$ to the second — or larger — power can be dropped).

$$\Delta F = GMm\left(\frac{1}{r^2} - \frac{1}{r^2(1+\frac{x}{r})^2}\right)$$

$$\Delta F = \frac{GMm}{r^2}\left(1 - \frac{1}{(1+\frac{x}{r})^2}\right)$$

$$\Delta F \approx \frac{GMm}{r^2}\left(1 - \left[1 - 2\frac{x}{r}\right]\right)$$

$$\Delta F \approx \frac{GMm}{r^2}\left(2\frac{x}{r}\right)$$

$$\Delta F \approx \frac{2GMmx}{r^3}$$

[1] The title of this worksheet comes from an album title.
[2] Yes, that **is** a technical term.

(e) Given that the tensile strength of human bones is about 10^9 dyne/cm^2 and a reasonable cross-section for a human is about 500 cm^2, what is the maximum force that a human can endure before being ripped apart?[3]

$$F_{\text{max}} = 10^9 \times 500 = 5 \times 10^{11} \text{ dyne}$$

(f) If the tidal force outside a black hole (ΔF) is larger than the maximum force human bones can withstand (F_{max}) then we say that a person will get "spaghettified" by the black hole (i.e., they will be gravitationally ripped apart before plunging into the black hole). By "outside" we mean beyond the "event horizon" (aka the edge, or the "point of no return") of the black hole. Anything (including light) that gets closer than this distance to the center of a black hole *must* free-fall to the center of the black hole (known as the "singularity" — a single point of infinite density).

However, if the tidal force from a black hole is low enough, then a human can plunge directly into a black hole in one piece. For this to be true, the tidal force from a black hole at its event horizon must be smaller than the maximum force human bones can withstand. The event horizon is, by definition, one Schwarzschild radius away from the center of a black hole. The Schwarzschild radius is given by:

$$r_S = \frac{2GM}{c^2}$$

Using this equation, derive an expression for the tidal force of a black hole *at its event horizon.*

$$\Delta F = \frac{2GMmx}{r_S^3}$$

$$\Delta F = 2GMmx \left(\frac{c^2}{2GM} \right)^3$$

$$\Delta F = \frac{mxc^6}{4G^2M^2}$$

(g) Set the above tidal force equal to the maximum force human bones can withstand and solve for M. This represents the *minimum* black hole mass that will not spaghettify Prof. Chiang and thus allow him to float right through the event horizon in one piece.

$$\Delta F = \frac{mxc^6}{4G^2M^2} = 5 \times 10^{11} \text{ dyne}$$

$$M = 7 \times 10^{-7} \frac{\sqrt{mx}c^3}{G}$$

`In cgs.`

(h) To order of magnitude, let's say the professor is about 1 m tall and 100 kg.[4] Calculate an actual value for the M you just solved for, in solar masses.

$$M = 7 \times 10^{-7} \frac{\sqrt{mx}c^3}{G}$$

[3] Well below this force, even though you wouldn't be ripped apart quite yet, you'd probably be in some pretty intense pain! Also note that this value of tensile strength is an underestimate for Prof. Chiang because he has "always been a big drinker of milk" and has "never once broken a bone".

[4] Actually, these numbers would be more accurate for an Oompa-Loompa version of Prof. Chiang.

$$M \approx 9 \times 10^{35} \text{ g} \approx 450 \text{ M}_{\odot}$$

`A 450 solar mass black hole is much bigger than black holes that come from the deaths of single, massive stars. However, there are black holes that are millions to billions of solar masses at the centers of most relatively large galaxies (like our own Milky Way). Thus, there are many, many black holes in the Universe that Prof. Chiang can jump into and still remain whole!`

2. **Tidal Forces and Mass Transfer**

Looking back at our expression for ΔF, let's say that the m represents a little piece of mass that is bound to a star of total mass M_*, the M represents the total mass of a nearby compact object (such as a black hole or a white dwarf), r is still the distance from the little piece of mass to the center of the compact object, and x is now the distance from the little piece of mass to the center of the star it's bound to (in other words, if the little piece of mass is at the surface of its star then x is the radius of the star). Given all this, ΔF becomes the difference between the gravitational forces on the little piece of mass (the gravity of the compact object tries to pull the piece off the star, while the star's gravity is trying to keep the piece bound to itself).

If this tidal force is larger than the gravitational force of the star that is trying to keep the little piece of mass bound, then the little piece of mass will get ripped off the star and become gravitationally bound to the compact object. We then say that the piece of mass has been "transferred" to the compact object. It will most likely orbit the compact object for awhile, get ripped into smaller and smaller pieces by the tidal forces on it as it orbits, get hotter and brighter as it gets ripped apart and as smaller pieces run into themselves while orbiting, and finally it will probably all crash down onto the surface of the compact object where we say it has now "accreted" onto the compact object.

(a) Using this information, solve for r_{Roche}, the Roche Lobe (aka Roche Limit, aka Lagrange Point 1, aka the mass transfer point, aka the point past which mass will transfer to the compact object), in terms of the density of the star, the density of the compact object, and the radius of the compact object (R).

$$F_{\text{grav}} < \Delta F$$

$$\frac{GmM_*}{x^2} < \frac{2GMmx}{r^3}$$

$$M_* < \frac{2Mx^3}{r^3}$$

$$r^3 < \frac{2Mx^3}{M_*}$$

$$r^3 < \frac{2R^3\rho}{\rho_*}$$

$$r_{\text{Roche}} = R\left(\frac{2\rho}{\rho_*}\right)^{1/3}$$

(b) Using your formula above, calculate is the Roche Limit (in solar radii) for a 0.7 solar mass red giant (which has a radius of about 200 $R_\odot$) in orbit around a 0.5 solar mass white dwarf (which has a radius of about 1 $R_\oplus$).

$$\frac{\rho}{\rho_*} = \frac{M}{R^3}\frac{R_*^3}{M_*} \approx \frac{0.5}{0.01^3}\frac{200^3}{0.7} \approx 5.7 \times 10^{12}$$

$$r_{\rm Roche} = 0.01 \text{ R}_\odot \left(2 \times 5.7 \times 10^{12}\right)^{1/3}$$

$$r_{\rm Roche} = 225 \text{ R}_\odot$$

(c) The Roche Limit of the Earth ($\rho_\oplus \approx 5.5$ g/cm^3 and $R_\oplus \approx 6.4 \times 10^8$ cm) for a human ($\rho \approx 1$ g/cm^3) is about 14,000 km (recall that this is the distance from the center of the Earth to the Roche Limit). We're having section on the surface of the Earth ($\sim$ 6400 km from the center of the Earth). Thus, we're all well within the Roche Limit of the Earth. Why haven't we all been ripped to pieces?[5]

`Humans (and most other objects on Earth) are held together by` *electro-static forces* `and` **NOT** *gravitational forces*. `Thus, the Roche Limit doesn't affect humans.`

3. **Gravitational Redshift**

Recall that earlier in the semester we talked about the Doppler Effect which related the line-of-sight (i.e., radial) velocity (v) to the observed wavelength of a photon (λ) and the actual emitted (i.e., rest) wavelength of a photon (λ_0) and the "redshift" of an object, defined by:

$$z \equiv \frac{v}{c} = \frac{\lambda - \lambda_0}{\lambda_0}$$

where the right hand side is only valid for velocities much less than the speed of light.

In General Relativity (GR) we can define a similar quantity, known as "gravitational redshift":

$$z_{\rm grav} = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2} - 1$$

where M is the mass of the black hole and r is the distance from the center of the black hole.

The details of gravitational redshift are outside the scope of this class, but basically you can think of photons as being affected by gravity – similar to a massive particle – when gravity is very strong, like near the edge of a black hole.

(a) Using the above equations, solve for the relative gravitational redshift of a photon in terms of its observed (λ) and emitted (λ_0) wavelengths (i.e., solve for λ/λ_0).

$$z_{\rm grav} = \frac{\lambda - \lambda_0}{\lambda_0} = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2} - 1$$

$$\frac{\lambda}{\lambda_0} - 1 = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2} - 1$$

$$\frac{\lambda}{\lambda_0} = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2}$$

[5]When Jeff was first asked this question as an undergrad in his planetary dynamics class he was very bothered for a solid day or two...

(b) Write down the relationship between frequency and wavelength and use it to get an expression for ν/ν_0.

$$\nu = \frac{c}{\lambda}$$

$$\frac{\nu}{\nu_0} = \left(1 - \frac{2GM}{rc^2}\right)^{1/2}$$

(c) Write down the relationship between energy and frequency and use it to get an expression for E/E_0.

$$E = h\nu$$

$$\frac{E}{E_0} = \left(1 - \frac{2GM}{rc^2}\right)^{1/2}$$

(d) Write down the relationship between the period of a light wave and its frequency and use it to get an expression for P/P_0.

$$P = \frac{1}{\nu}$$

$$\frac{P}{P_0} = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2}$$

(e) Now that we have a bunch of relationships between emitted and observed values, let's find out what happens right at the event horizon of a black hole. To do this set the radius, r, in all of the above expressions equal to r_S.

$$\frac{\lambda}{\lambda_0} \to \infty$$

$$\frac{\nu}{\nu_0} \to 0$$

$$\frac{E}{E_0} \to 0$$

$$\frac{P}{P_0} \to \infty$$

`This means that the wavelength of a photon emitted from from the event horizon of a black hole gets redshifted to infinity, which means that both its frequency and energy get redshifted to 0 and that its period slows down to infinitely long.`

(f) Using what you've calculated above, describe what observers on Earth would see as a digital alarm clock that showed hours, minutes, and seconds in huge blue numbers fell through the event horizon of a black hole (assuming we had a big enough and sensitive enough telescope to resolve the numbers!!). Assume that the black hole is large enough such that the clock is not spaghettified before falling into the black hole.

`As the clock gets extremely close to the event horizon, the physical properties of it tend toward our extreme values calculated above.  Thus the light from` the numbers on the clock would be redshifted from blue, to greenish, then to red, and eventually out of the visible spectrum into infrared and radio wavelengths and finally to infinitely long wavelengths. This corresponds to the frequency of the observed light approaching zero.

Another thing to remember is the diffraction limit, $\theta \approx \lambda/D$. As the wavelength gets stretched to infinity, the diffraction limit of a given telescope increases and so the numbers on the clock actually get more and more difficult to resolve. Thus we'd need a bigger and bigger telescope to be able to actually read the time off the clock.

Also, if we could still detect the numbers on the clock at these extremely long wavelengths (possibly using a crazy big radio telescope), we would see that the numbers themselves are getting dimmer since the emitted energy is being redshifted toward zero.

Finally, we would see the clock itself slow down (again assuming we could actually somehow detect the numbers). Time (due to GR effects), as we're reading it off the clock face, is slowing down more and more as it approaches the event horizon and will eventually seem to stop completely as it crosses the horizon. Keep in mind, it only *seems* to stop completely, it is actually getting arbitrarily close to completely stopping (in other words the time between the seconds changing is approaching infinity). Thus, if we had an ever more accurate alarm clock, we would see that when the seconds seem to ''stop completely'' the milliseconds are still ticking away (getting ever slower however), and eventually they would seem to stop but the nanoseconds would still be ticking (again, slower and slower), and so on and so forth. Therefore, we never actually see the alarm clock disappear across the event horizon --- it appears to us as frozen right at the edge of the black hole!

Aren't black holes and GR weird?!?!