

Ay 7A - Fall 2009
Section Worksheet 3
Let There Be Light!

1. Deriving Stefan-Boltzmann from Planck

In this problem we will derive the fact that the bolometric flux of a black body is equal to σT^4 .

- (a) Let's start by integrating the Planck function, $B_\lambda(T)$, over *all* wavelengths. Recall that $B_\lambda(T)$ is given by

$$B_\lambda(T) = \frac{2hc^2/\lambda^5}{e^{hc/\lambda kT} - 1} \quad (1)$$

You may use the following formula for the integration,

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15} \quad (2)$$

First, make a change of variables as follows,

$$x = \frac{hc}{\lambda kT}, \quad dx = -\frac{hc}{kT} \frac{1}{\lambda^2} d\lambda. \quad (3)$$

Then the integral of $B_\lambda(T)$ with respect to $d\lambda$ from 0 to ∞ becomes

$$\begin{aligned} \int_0^\infty B_\lambda(T) d\lambda &= \int_0^\infty \frac{2hc^2/\lambda^5}{e^{hc/\lambda kT} - 1} d\lambda \\ &= \int_\infty^0 2hc^2 \left(\frac{xkT}{hc}\right)^3 \frac{1}{e^x - 1} \left(-\frac{kT}{hc}\right) dx \\ &= 2hc^2 \left(\frac{kT}{hc}\right)^4 \int_0^\infty \frac{x^3}{e^x - 1} \\ &= \frac{2\pi^4 k^4 T^4}{15h^3 c^2} \end{aligned} \quad (4)$$

- (b) Noticing that $B_\lambda(T)$ is a particular form of specific intensity and the integration over all solid angles yields a factor of π , what is the bolometric *flux* (i.e. the *flux* integrated over all wavelengths) of a black body? (This should recover the Stephan-Boltzmann equation, $F = \sigma T^4$.) Express σ in terms of fundamental constants (you **do not** have to plug in the values of the constants).

Since the Planck function is a form of specific intensity, the bolometric flux is just

$$F = \int \int_0^\infty B_\lambda(T) d\lambda d\Omega \quad (5)$$

where Ω is solid angle (in steradians). So,

$$\begin{aligned} F &= \int \frac{2\pi^4 k^4 T^4}{15h^3 c^2} d\Omega \\ &= \frac{2\pi^4 k^4 T^4}{15h^3 c^2} \int d\Omega \\ &= \frac{2\pi^5 k^4 T^4}{15h^3 c^2} \\ &= \sigma T^4 \quad \left(\text{where } \sigma = \frac{2\pi^5 k^4}{15h^3 c^2}\right) \end{aligned} \quad (6)$$

- (c) (**optional**) In class, we learned $B_\lambda \neq B_\nu$ because equal intervals in λ don't necessarily imply equal intervals in ν . However, $B_\lambda d\lambda = B_\nu d\nu$. Show that this is true.

From $\lambda\nu = c$, we get

$$\nu = \frac{c}{\lambda} \text{ and } d\nu = -\frac{c}{\lambda^2} d\lambda. \quad (7)$$

Note that the minus sign in the relation between $d\nu$ and $d\lambda$ will cancel out because when we integrate B_ν over a small interval $[\nu_1, \nu_2]$ (where $\nu_1 < \nu_2$), it corresponds to the integration of B_λ over $[\lambda_1, \lambda_2]$ but $\lambda_1 > \lambda_2$. Plug Eq. (7) into $B_\nu d\nu$, then it becomes

$$\begin{aligned} B_\nu d\nu &= \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1} d\nu \\ &= \frac{2h(c/\lambda)^3/c^2}{e^{hc/\lambda kT} - 1} \frac{c}{\lambda^2} d\lambda \\ &= \frac{2hc^2/\lambda^5}{e^{hc/\lambda kT} - 1} d\lambda \\ &= B_\lambda d\lambda \end{aligned} \quad (8)$$

2. Measuring the Radius of a Star

Even nearby stars are far away from us, so we see a star as a point source. However, using observed quantities of a star, we can measure its radius. Here, we will estimate the radius of a star assuming the star is a perfect black body and is radiating perfectly isotropically.

- (a) You observe a star and measure its parallax angle, θ , in arcseconds. What is the distance, d , to this star from Earth?

The distance can be easily expressed in parsecs, since:

$$d \text{ (in pc)} = \frac{1}{\theta \text{ (in arcsec)}} \quad (9)$$

- (b) Also, you recognize that its temperature is T (from the star's spectrum). What is the bolometric flux of this star (F_*) at its surface? What is the luminosity of the star (L_*)?

From Stefan-Boltzmann's law, $F_* = \sigma T^4$. Flux is luminosity per area, so the luminosity of the star is simply $L_* = 4\pi R_*^2 \sigma T^4$, where R_* is the radius of the star.

- (c) On Earth, we measure the bolometric flux received from the star to be F . Using this measurement (and the above results) estimate the size of the star.

The flux measured on Earth is simply given by

$$F = \frac{L_*}{4\pi d^2} = \frac{R_*^2 \sigma T^4}{d^2} \quad (10)$$

Thus, $R_* = \sqrt{\frac{F d}{\sigma T^2}}$

- (d) Finally let's plug some real numbers into the equations we got. Let's say the star's surface temperature is 8000 K (from its spectrum), its bolometric (i.e. integrated over all wavelengths) flux observed from Earth is 7.38×10^{-5} erg/s/cm², and that the star has a parallax of 0.25". Using these numbers, what is the size of the star? Express the radius in solar units. (Recall that $R_{\odot} = 6.96 \times 10^{10}$ cm, 1 pc = 3.09×10^{18} cm, and $\sigma = 5.67 \times 10^{-5}$ erg/cm²/s/K⁴.)

Plugging in the given values, and solving for R_* we obtain d is 4 pc and then

$$R_* = 2.1 \times 10^{11} \text{ cm} \approx 3R_{\odot}$$

3. You're (sort of) a Black Body

In lecture we heard that the Cosmic Microwave Background radiation which permeates the Universe is the closest thing to an exact black body that we know of. However, oftentimes we can approximate stars as black bodies. We can even, to a *very rough* approximation, say that humans are black bodies.

- (a) Assuming that *your* body is a black body, what is the peak wavelength of electromagnetic (EM) radiation being emitted by you? Recall that $[K] = \frac{5}{9} ([^{\circ}F] + 459.67)$. Human body temperature is 98.6° F. Using the equation above, this converts to about 310 K. Using Wien's law, $\lambda_{\text{peak}} T = 0.29$ cm K, we get that healthy humans should have a peak wavelength of about 9.35 micrometers (μm or microns).
- (b) What part of the EM spectrum is this?
Infrared (IR).
- (c) How does this relate to how night-vision goggles work?
When it's dark (at night, for example) there's no visible light to reflect off of stuff so we can't see anything. However, warm-blooded creatures (like humans) still give off quite a bit of IR radiation. Thus there are IR detectors in night-vision goggles that are tuned to about 10 microns --- the wavelength at which radiation (heat) from a human body peaks!
- (d) The solar surface is about 6000 K and peaks in the visible part of the EM spectra (around the colors yellow and green, to be more precise). When people wear a yellow or green shirt their shirt is clearly not as hot as the surface of the Sun (no matter how hot you might think they look!), so why is their shirt the color that it is?
Shirts, along with any other inanimate object that doesn't produce its own energy, get their color by reflecting certain colors and absorbing others, **NOT** by radiating energy (black body or otherwise).

4. Arc Lamps!!!!

We will look at examples of black body and emission-line spectra with a diffraction grating — an optical component which disperses and diffracts light, allowing us to “see” spectral features.

PLEASE only hold the diffraction gratings by the white border; the oils on your hands and fingers will clog the tiny slits on the clear part, thus rendering the grating useless!

- (a) Black Body Spectrum

First, look through the diffraction grating (make sure to keep it oriented so that you can read the words on the white part of the grating) at the ordinary incandescent light-bulb. You should see a horizontal band of color some distance off to the side of the object.

- i. Sketch this band as best you can in the box, indicating the order and sizes of the regions of the different colors and label the color of each region.



The band should be a continuous rainbow of colors going from blue on the left, to red on the right.

- ii. Translate this sketch into a rough diagram of wavelength versus flux density (i.e. a spectrum). Make sure that your spectrum has wavelength increasing to the right since that's how the diffraction gratings are designed.



The spectrum should be a flat, horizontal line since each color is approximately as bright as all the other colors.

- iii. Why does the bulb looks white?

The bulb appears white to our eyes since we cannot break the light into its component wavelengths like the grating can. Thus, we see a combination of all the colors and that appears as white.

(b) Emission Spectra

Next, we will look at the spectra of a couple different elements using the arc lamp, a simple device that passes an electric current through a gas-filled tube. (This is how a neon sign works!) Sometimes seeing these lines can be a bit tricky — if you have difficulty, try holding the lens closer to your eye and/or getting closer to the source.

In the area below, write the name of the element and make a quick sketch of the positions and colors of the brightest spectral features. (You can also just describe the spectral features qualitatively: “lots of red lines”, etc.) Also note the overall color of the glowing tube to your eye (without the grating).

Element:

Overall Color:



The exact picture depends on the element, but there should only be certain discreet vertical lines of various colors, with bluer colors on the left and redder colors on the right.

(c) The Emission Spectrum of Hydrogen

As mentioned in class, hydrogen is the most abundant element in the Universe and also the easiest to understand.

Element: H

Overall Color:

(HINT: There should be four lines that you can see, but one is very faint!)



There should be 4 vertical lines. A red one toward the right, a light blue one left of center, a darker blue one further left, and a purple one on the far left.

- i. Translate this into a rough spectrum.



There should be four narrow spikes that correspond to the four vertical lines in the drawing above.

- ii. A diagram of the hydrogen atom is shown in Figure 1. The numbered circles designate some of the inner energy levels. As it turns out, the spectral lines you saw moments ago correspond to the transitions $3 \rightarrow 2$, $4 \rightarrow 2$, $5 \rightarrow 2$, and $6 \rightarrow 2$. These are the Balmer lines ($H\alpha$, $H\beta$, $H\gamma$, and $H\delta$, respectively). Which transition corresponds to which spectral line? (Label the lines in your spectrum above).

The lines, from right to left, follow the Greek alphabet. $H\alpha$ is the red line, $H\beta$ is the light blue line, $H\gamma$ is the dark blue line, and $H\delta$ is the purple line. We know this since the lowest energy transition is $3 \rightarrow 2$ (aka $H\alpha$) and the red line has the longest wavelength (and thus the lowest energy).

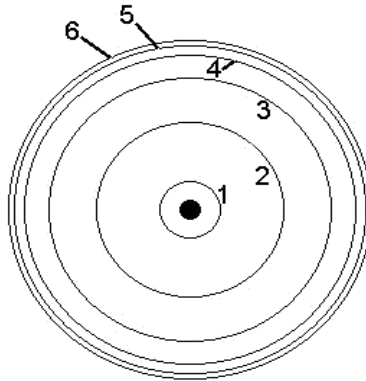


Figure 1: A simplified hydrogen atom.

- iii. Why don't you see any lines from transitions to the ground state (level 1) in the spectrum? These are the Lyman lines.
All of the Lyman lines are in the UV. They are too high energy to see in visible light.
- iv. Why don't you see any lines from transitions to level 3? These are the Paschen lines.
All of the Paschen lines are in the IR. They are too low energy to see in visible light.