

# Ay 7A - Fall 2009

## Section Worksheet 2

### 1. Escape Velocity

- (a) A planet is orbiting around a star. What is the total orbital energy of the planet? (i.e. Total Energy = Potential Energy + Kinetic Energy). To just barely escape the orbit (and the gravitational attraction of the star), what is the minimum total energy the planet should have?

Potential energy is given by  $U = -G\frac{Mm}{r}$  and kinetic energy is given by  $K = \frac{1}{2}mv^2$  where  $M$  is the stellar mass,  $m$  is the planet's mass and  $v$  is the speed of the planet orbiting around the star. Then total energy is

$$E = U + K = -G\frac{Mm}{r} + \frac{1}{2}mv^2 \quad (1)$$

To just barely escape the orbit, the velocity,  $v$ , of a planet at infinity (i.e.  $r \rightarrow \infty$ ) becomes 0. So,  $v \rightarrow 0$  and  $r \rightarrow \infty$ , thus  $E \rightarrow 0$  at infinity and by energy conservation, the minimum energy the escaping planet needs to have is 0. Note that the energy of the *bound* orbit is always negative.

- (b) Using the above result, what is the escape velocity of the planet from the star? (The **escape velocity** is defined as the minimum velocity an object needs to escape the gravitational pull of another object.)

$$E = 0 = -G\frac{Mm}{r} + \frac{1}{2}mv^2$$
$$G\frac{Mm}{r} = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{2GM}{r}} \quad (2)$$

- (c) Calculate the escape velocity of an object from the surface of the Earth (only considering the Earth's gravitational pull).

$$v_{esc} = \sqrt{\frac{2GM_{\oplus}}{R_{\oplus}}} \approx 11.2 \text{ km/s}$$

### 2. Geosynchronous Satellite

- (a) Communications and weather satellites are often placed in geosynchronous "parking" orbits above earth. These are orbits where satellites can remain fixed above a specific point on the surface of Earth. At what altitude must these satellites be located?

The orbital period of a geosynchronous satellite must be the same as Earth's sidereal rotation period: the time it takes stars to "orbit" the Earth. This value is  $P = 23 \text{ h } 56 \text{ m} = 8.616 \times 10^4 \text{ s}$ . From Eq. (2.37) in CO (i.e. Kepler's Third Law),  $a = 4.21 \times 10^9 \text{ cm}$ , implying an altitude of  $h = a - R_{\oplus} = 3.58 \times 10^9 \text{ cm} = 5.6 R_{\oplus}$ .

- (b) Is it possible for a satellite in a geosynchronous orbit to remain “parked” over any location on the surface of Earth? Why or why not?

A geosynchronous satellite must be “‘parked’” over the equator and orbiting in the direction of Earth’s rotation. This is because the center of the satellite’s orbit is the center of mass of the Earth-satellite system (which is essentially Earth’s center).

- (c) Jeff has DirecTV at his house in the Berkeley Hills ( $\sim 38^\circ$  north latitude). How many degrees away from the zenith (which is what astronomers call “directly overhead”) should he point his satellite dish in order to be aimed at the nearest place a geosynchronous satellite could be?

As you see in Figure 1, we want  $\beta$ . Also from Figure 1 we see that  $R$  is the radius of the Earth,  $a$  is the distance from the center of the Earth to the geosynchronous satellite (or the semi-major axis of its orbit, which we calculated in the previous part), and  $\alpha$  is the angle from the equator to Berkeley (i.e.  $\sim 38^\circ$ ). Using law of cosines we have  $c = (R^2 + a^2 - 2Ra \cos \alpha)^{1/2}$ , and law of sines gives us  $\frac{\sin \alpha}{c} = \frac{\sin(\pi - \beta)}{a}$ . Also, using the “‘sine of a difference formula’” we get  $\sin(\pi - \beta) = \sin \pi \cos \beta - \cos \pi \sin \beta = \sin \beta$ . Put this all together and we get:

$$\sin(\pi - \beta) = \sin \beta = \frac{a}{c} \sin \alpha = a \sin \alpha (R^2 + a^2 - 2Ra \cos \alpha)^{-1/2}. \quad (3)$$

Thus,  $\boxed{\beta \approx 44^\circ}$ .

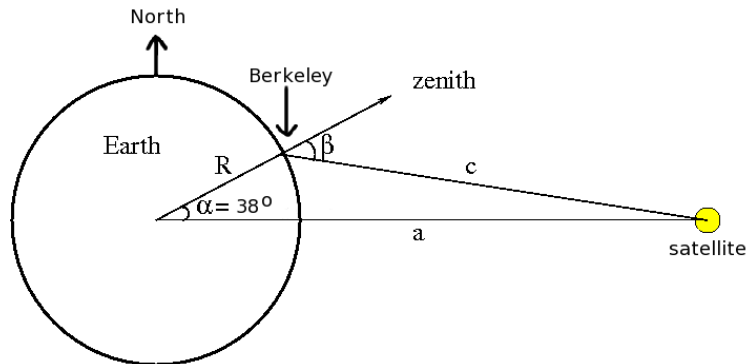


Figure 1: Pointing at the nearest geosynchronous satellite from Berkeley.

### 3. Hohmann Transfer

Orbits are changed by single or multiple thrusts of the rocket engines. The simplest maneuver is a single thrust applied in the orbital plane that does not change the direction of the angular momentum but does change the eccentricity and energy simultaneously. The most economical method of interplanetary transfer consists of moving from one circular heliocentric (sun-oriented motion) orbit to another in the same plane. Earth and Mars represent such a system reasonably well, and a Hohmann transfer represents the path of minimum total energy expenditure. Two engine burns are required: (1) the first burn injects the spacecraft from the circular Earth orbit to an elliptical transfer orbit that intersects Mars’ orbit; (2) the second

burn transfers the spacecraft from the elliptical orbit into Mars' orbit. We will calculate the velocity changes needed for a Hohmann transfer in this problem. Here we are considering only the gravitational attraction of the sun and not that of Earth and Mars (Figure 2). For this problem you don't have to plug in numbers if you don't want to (until the last part), but you should be able to solve everything in terms of the mass of the Sun, the Sun-Earth distance, the Sun-Mars distance, and the gravitational constant,  $G$ .

- (a) The spacecraft is moving with  $\mathbf{v}_1$  in the orbit of the Earth around the Sun (the Sun-Earth distance is  $r_1$  in Figure 2). What is the velocity needed to "kick" it into an elliptical transfer orbit that can reach Mars' orbit (the Sun-Mars distance is  $r_2$  in Figure 2)? You may start with the total orbital energy equation you used in 'Escape Velocity' problem.

Recall the equation of the energy of the 2-body problem:

$$E = -\frac{k}{2a}, \text{ where } k = GM_{\odot}m. \quad (4)$$

For a circular path around the sun at  $r_1$ , this becomes

$$E = \frac{-k}{2r_1} = \frac{1}{2}mv_1^2 - \frac{k}{r_1}. \quad (5)$$

We solve above equation for  $v_1$ :

$$v_1 = \sqrt{\frac{k}{mr_1}}. \quad (6)$$

We denote the semimajor axis of the transfer ellipse by  $a_t$ , then  $2a_t = r_1 + r_2$ . If we calculate the energy at the perihelion for the transfer ellipse, we have

$$E_t = \frac{-k}{r_1 + r_2} = \frac{1}{2}mv_{t1}^2 - \frac{k}{r_1} \quad (7)$$

where  $v_{t1}$  is the speed at perihelion for the transfer ellipse. The direction of  $v_{t1}$  is along  $\mathbf{v}_1$  in Figure 2. Solving the above equation for  $v_{t1}$  gives

$$v_{t1} = \sqrt{\frac{2k}{mr_1} \frac{r_2}{r_1 + r_2}} \quad (8)$$

and then the change in speed,  $\Delta v_1$ , needed is just  $\Delta v_1 = v_{t1} - v_1$ .

- (b) Similarly, for the transfer from the ellipse to the circular orbit of radius  $r_2$ , what is the transfer speed needed?

Similarly, for the transfer from the ellipse to the circular orbit of radius  $r_2$ , we have  $\Delta v_2 = v_2 - v_{t2}$  where

$$v_2 = \sqrt{\frac{k}{mr_2}} \text{ and } v_{t2} = \sqrt{\frac{2}{m} \left( E_t + \frac{k}{r_2} \right)} = \sqrt{\frac{2k}{mr_2} \frac{r_1}{r_1 + r_2}} \quad (9)$$

The direction of  $v_{t2}$  is along  $\mathbf{v}_2$  in Figure 2.

(c) What is the total time required to make the transfer?

The total time required to make the transfer  $T_t$  is a half-period of the transfer orbit. From Kepler's law, we have

$$T_t = \frac{P}{2} = \pi \sqrt{\frac{m}{k}} a_t^{3/2}. \quad (10)$$

(d) Plug in the appropriate constants and calculate the time needed for a spacecraft to make a Hohmann transfer from Earth to Mars. Assume both planets are in coplanar orbits.

Using the result in (c),

$$\frac{m}{k} = \frac{m}{GmM_\odot} \approx 7.53 \times 10^{-27} \text{ s}^2/\text{cm}^3 \quad (11)$$

$$2a_t = r_{\text{Earth-Sun}} + r_{\text{Mars-Sun}} \approx 3.78 \times 10^{13} \text{ cm} \quad (12)$$

$$T_t \approx 2.24 \times 10^7 \text{ s} \approx 259 \text{ days} \quad (13)$$

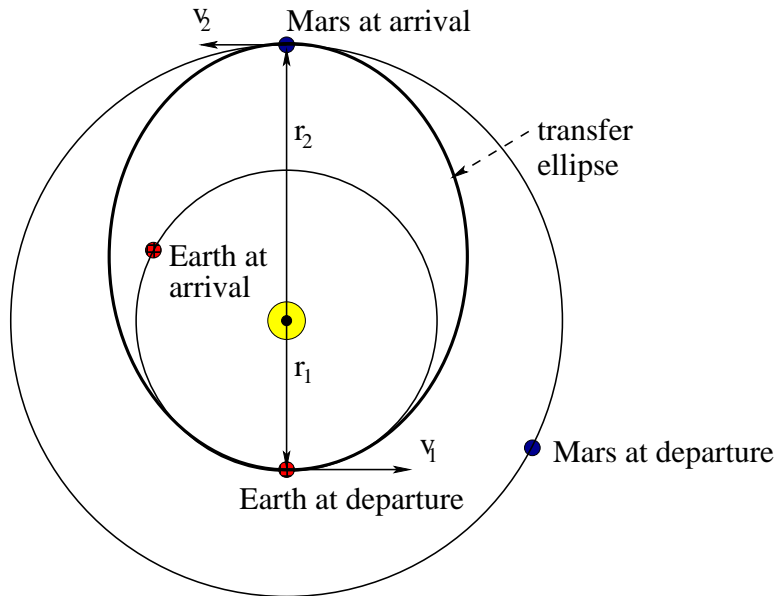


Figure 2: A diagram of Hohmann Transfer

#### 4. (optional) Homogeneous Sphere

(a) Imagine a sphere has constant density  $\rho$ . What is the mass of the sphere (as a function of its radius,  $r$ )?

$$M(r) = \int_0^r dr \, 4\pi r^2 \rho = \frac{4\pi r^3}{3} \rho \quad (14)$$

- (b) Using the Newton's first<sup>1</sup> and second theorem<sup>2</sup>, get the equation of the motion in terms of  $\rho$  and  $r$  if a test particle is released from rest at radius  $r$  in the gravitational field of a homogeneous sphere.

$$\frac{d^2r}{dt^2} = -\frac{GM(r)}{r^2} = -\frac{4\pi G\rho}{3}r \quad (15)$$

- (c) Using Cartesian coordinates  $(x, y)$  defined by  $x = r \cos \theta$ ,  $y = r \sin \theta$  and defining  $\Omega^2 = \frac{4\pi G\rho}{3}$ , rewrite the equations of motion of  $x$  and  $y$  in terms of  $\Omega$ .

$$\ddot{x} = -\Omega^2 x, \quad \ddot{y} = -\Omega^2 y \quad (16)$$

- (d) What is the form of the equations of motion? Write down the general solution of the equations of motion.

The form of the equations of motion is simple harmonic oscillations. The solutions are

$$x = X \cos(\Omega t + \epsilon_x), \quad y = Y \sin(\Omega t + \epsilon_y) \quad (17)$$

where  $X$ ,  $Y$ ,  $\epsilon_x$ , and  $\epsilon_y$  are arbitrary constants and  $\Omega$  is the frequency of oscillations.

- (e) What is the shape of the orbits of the test particle? Are the orbits closed? If we assume the mass is concentrated in a point, where is the point located? Compare this to Keplerian motion we derived in class<sup>3</sup>.

The orbits form ellipses. That means the orbits are closed since the periods of the oscillations in  $x$  and  $y$  are identical. The point mass would be located at the *center* of the ellipses in this case, while for Keplerian motion, the point mass is located at one of the foci of the ellipses.

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<sup>1</sup>Newton's first theorem: A body that is inside a spherical shell of matter experiences no net gravitational force from that shell.

<sup>2</sup>Newton's second theorem: The gravitational force on a body that lies outside a spherical shell of matter is the same as it would be if all the shell's matter were concentrated into a point at its center.

<sup>3</sup>Note that Keplerian motion refers to motion in the gravitational field due to a point mass  $M$  corresponding to the potential  $-\frac{GM}{r}$ .