# Ay 7A – Fall 2009
# Section Worksheet 7
# Scaling the Walls

1. **Calculus for Dummies!**

   In lecture we're going to see more and more differential equations. In general, you won't be expected to solve them exactly, but you will be expected to analyze the properties of their solutions in a simplified way. This often involves doing something called "order of magnitude (OOM) differentiation".[1]

   (a) Let's start off with an example. Say that the velocity of a particle is given by $v = \alpha t^{3/2}$ where $\alpha$ is a constant, and we want to find the scaling of position with time. First write down the equation in the form of a differential equation for $x$, the position.

   (b) Next, we are going to say that $dx \sim x$ and $dt \sim t$. Now it should be easy to show the scaling of $x$ with $t$...what is it?

   (c) Now what was the point of that? I mean, this is a simple differential equation, you can integrate it. So go ahead, do the integral, and what do you get? How does it compare to your scaling relationship?

   (d) Now back to astronomy, the equation for hydrostatic equilibrium (HSE) is

   $$\frac{dP}{dr} = -\rho g$$

   where $P$ is the pressure at height $r$ from some reference height (perhaps the ground or the center of a star), $\rho$ is the density (as a function of $r$) and $g$ is the gravitational acceleration (also as a function of $r$).

   We'd like to get an idea of how a star's central pressure scales with other quantities. Note that we aren't solving for an exact pressure right now, we just want some general ideas about how a star's physical properties depend on its other properties.

   Approximate $dP/dr$ as $\Delta P/\Delta r$ and rewrite the HSE equation in an approximate form, in terms of the star's central pressure ($P_c$), radius ($R$), density, and gravitational acceleration. Be careful with that minus sign...

[1] We actually sorta did a couple examples of this already in PS 3.

(e) Using Newton's Laws and evaluating at the surface of the star we get:

$$F = ma = mg = G\frac{mM}{R^2}$$

and thus $g \propto \frac{M}{R^2}$ where $M$ is the total mass of the star and $R$ is again the radius of the star. We can also take $\rho$ to be some average density, which is simply mass divided by volume, so $\rho \propto \frac{M}{R^3}$.

Rewrite our HSE equation as a proportionality only involving $P_c$, $M$, and $R$.

(f) Now we can look at how the central pressure, $P_c$, scales with various quantities.

i. If you increase the mass of the star, what happens to $P_c$?

ii. If you increase the radius, what happens to $P_c$?

iii. If you increase the average density what happens to $P_c$?

(g) The real beauty of scaling is that we can use our knowledge of the Sun ($M_\odot$, $\rho_\odot$, $R_\odot$, etc.) to scale up or down to the central pressure of another star. The central pressure in the Sun is about $2.34 \times 10^{17}$ dyne/cm$^2$. What is the central pressure of Sirius A which has $R \approx 1.7\ R_\odot$ and $M \approx 2.0\ M_\odot$?

2. **"I like you temporarily!"**[2]

Now we're going to do a little demo that will hopefully give you a slightly better insight into what we mean when we talk about hydrostatic equilibrium.

(a) I have a balloon and a small hair dryer. I turn the hair dryer on using its "low" setting and place the balloon above it. What happens? Why? (HINT: Think about the forces acting on the balloon.)

(b) How do things change when I turn the hair dryer to its "high" setting? Why?

[2]This quote comes from a recent movie that is somewhat related to the following demo.

(c) Now I will replace the balloon with a slightly heavier balloon and keep the hair dryer on "high". What changes in this case? Why?

(d) Finally, I will turn the hair dryer back to its "low" setting. What changes now? Why?

(e) This has been a decent analogy for hydrostatic equilibrium in stars. Let's make the analogy a bit more explicit now. What do the two different settings on the hair dryer represent? What does the Earth's gravity represent? What do the two balloons represent?

3. **One Ring To Rule Them All**

For our next demo, we are going to observe the effect known as "limb brightening". In lecture we discussed the opposite effect, "limb darkening", quite a bit (see Figure 1).

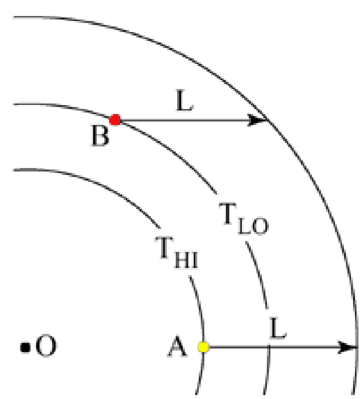


Figure 1: Limb Darkening in the Sun: When looking at the Sun in visible light, $\tau = 1$ is reached after you look a distance $L$ through the Sun's atmosphere. When you look toward the center of the solar disk, this distance corresponds to point $A$. When you look at the edge (or "limb") of the Sun, this distance corresponds to point $B$, which you can see is at a larger distance from the center of the Sun (point $O$) than point $A$. The Sun's temperature decreases with radius in this region and thus the temperature at point $A$ ($T_{\rm HI}$) is greater than the temperature at point $B$ ($T_{\rm LO}$). Since $F \propto T^4$, the higher temperature areas will appear brighter than the lower temperature areas and so the limb of the Sun looks darker than the center of the solar disk.

Again using optical depth arguments, we will explain the phenomenon of "limb brightening". Let's turn off the lights...

(a) Here we have a (collapsed) Hoberman Sphere, to which we have attached some lights. The lights are roughly uniformly distributed on the surface of the sphere. Close one eye to simulate the lack of depth perception that astronomers have when observing objects on the celestial sphere. Can you see through the sphere? What do astronomers call such an object (in terms of its optical depth)?

(b) Does any part of the sphere look any brighter than any other part? Or, what shape do you see?

(c) Why don't we see limb darkening here (like we do for the Sun which is certainly also optically thick)?

(d) Now I will expand the sphere. Can you see through the sphere? What do astronomers call such an object (in terms of its optical depth)?

(e) Does any part of the sphere look any brighter than any other part? Or, what shape do you see?

(f) Why do we see limb brightening here?

(g) If I rotate the sphere (allowing you to observe it from different viewing angles), do your observations change?

Lots of objects in astronomy are spherical and optically thin and thus appear as rings. One common example are planetary nebulae (which we'll learn more about later this semester). Basically they are evolved low-mass stars that eject (or "burp" off) their outer layers of gas. There are some pictures of these objects around the room, including one of the most famous and easy to find in a small telescope, the Ring Nebula.

Also, expanding supernovae ejecta (the explosive deaths of stars) are usually fairly spherical and become optically thin very soon after the explosion. Probably the best-studied supernova of all time, SN 1987A, appears quite ring-like today.

(h) If many astronomical objects appear to us as rings. How do we know that many of them are in fact optically thin spheres? (HINT: Both planetary nebulae and supernova ejecta are *expanding* optically thin spheres.)