Ay 7A – Fall 2009
Section Worksheet 7
Scaling the Walls

1. **Calculus for Dummies!**

   In lecture we're going to see more and more differential equations. In general, you won't be expected to solve them exactly, but you will be expected to analyze the properties of their solutions in a simplified way. This often involves doing something called "order of magnitude (OOM) differentiation".[1]

   (a) Let's start off with an example. Say that the velocity of a particle is given by $v = \alpha t^{3/2}$ where α is a constant, and we want to find the scaling of position with time. First write down the equation in the form of a differential equation for x, the position.
   `From physics we know that` $v \equiv dx/dt$ `and thus` $dx/dt = \alpha t^{3/2}$.

   (b) Next, we are going to say that $dx \sim x$ and $dt \sim t$. Now it should be easy to show the scaling of x with t...what is it?
   `OK, then` $x/t = \alpha t^{3/2}$ `or` $x = \alpha t^{5/2}$`, and finally` $x \propto t^{5/2}$.

   (c) Now what was the point of that? I mean, this is a simple differential equation, you can integrate it. So go ahead, do the integral, and what do you get? How does it compare to your scaling relationship?
   `We can integrate the expression directly to obtain` $x = \alpha(2/5)t^{5/2} + C$ `where` C `is some constant of integration. We again see that` $x \propto t^{5/2}$`, so our quick and dirty method above to solve the differential equation gave us the correct power law!`

   (d) Now back to astronomy, the equation for hydrostatic equilibrium (HSE) is

   $$\frac{dP}{dr} = -\rho g$$

   where P is the pressure at height r from some reference height (perhaps the ground or the center of a star), ρ is the density (as a function of r) and g is the gravitational acceleration (also as a function of r).

   We'd like to get an idea of how a star's central pressure scales with other quantities. Note that we aren't solving for an exact pressure right now, we just want some general ideas about how a star's physical properties depend on its other properties.

   Approximate dP/dr as $\Delta P/\Delta r$ and rewrite the HSE equation in an approximate form, in terms of the star's central pressure (P_c), radius (R), density, and gravitational acceleration. Be careful with that minus sign...

   $$\frac{dP}{dr} \approx \frac{\Delta P}{\Delta r} \approx -\rho g$$
   $$\frac{P_c - P_{\rm surf}}{0 - R} \approx -\rho g$$
   $$\frac{P_c}{-R} \approx -\rho g$$
   $$\frac{P_c}{R} \approx \rho g$$

   `where we have integrated from the center of the star (`$r = 0$`) to the surface of the star (`$r = R$`) and we have noticed that the pressure of a star at its surface (`$P_{\rm surf}$`) is much, much smaller than the central pressure.`

[1] We actually sorta did a couple examples of this already in PS 3.

(e) Using Newton's Laws and evaluating at the surface of the star we get:

$$F = ma = mg = G\frac{mM}{R^2}$$

and thus $g \propto \frac{M}{R^2}$ where M is the total mass of the star and R is again the radius of the star. We can also take ρ to be some average density, which is simply mass divided by volume, so $\rho \propto \frac{M}{R^3}$.
Rewrite our HSE equation as a proportionality only involving P_c, M, and R.

$$\frac{P_c}{R} \propto \rho g \propto \frac{M}{R^3}\frac{M}{R^2}$$
$$P_c \propto \frac{M^2}{R^4} \propto \frac{M\rho}{R}$$

(f) Now we can look at how the central pressure, P_c, scales with various quantities.

i. If you increase the mass of the star, what happens to P_c? `It increases.`

ii. If you increase the radius, what happens to P_c? `It decreases.`

iii. If you increase the average density what happens to P_c? `It increases.`

(g) The real beauty of scaling is that we can use our knowledge of the sun ($M_\odot$, $\rho_\odot$, $R_\odot$, etc.) to scale up or down to the central pressure of another star. The central pressure in the Sun is about 2.34×10^{17} dyne/cm^2. What is the central pressure of Sirius A which has $R \approx 1.7\ R_\odot$ and $M \approx 2.0\ M_\odot$?

This is the punchline, and the whole point of doing these OOM integration scaling relationships. We found out already that $P_c \propto \frac{M^2}{R^4}$ in an earlier part. So if Sirius is 1.7 times larger and 2.0 times more massive than the Sun, the central pressure should be

$$\frac{2.0^2}{1.7^4} \approx 0.49$$

times as big. In other words, the central pressure of Sirius is about $0.49P_c \approx 1.1 \times 10^{17}$ dyne/cm^2. Isn't this a powerful tool we have here?!?!?!

2. **"I like you temporarily!"**[2]

Now we're going to do a little demo that will hopefully give you a slightly better insight into what we mean when we talk about hydrostatic equilibrium.

(a) I have a balloon and a small hair dryer. I turn the hair dryer on using its "low" setting and place the balloon above it. What happens? Why? (HINT: Think about the forces acting on the balloon.)

The balloon floats some distance above the hair dryer. The downward force of gravity on the balloon (which is more massive than the air inside and around it) is exactly balanced by the upward force of the hair dryer's blowing air (i.e., the pressure of the air from the hair dryer). This pressure decreases with distance away from the end of the hair dryer due to the air ''spreading out'' and not being perfectly collimated.

(b) How do things change when I turn the hair dryer to its "high" setting? Why?

The pressure from the hair dryer is increased thus its upward force (at a given distance from the end of the hair dryer) is increased. Therefore, to get back into equilibrium, the balloon must move higher (i.e., farther from the end of the hair dyer) such that it ''feels'' the same effective pressure as it did before (to balance the same downward force of gravity).

[2]This quote comes from a recent movie that is somewhat related to the following demo.

(c) Now I will replace the balloon with a slightly heavier balloon and keep the hair dryer on "high". What changes in this case? Why?

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Now the balloon floats down toward the end of the hair dryer, where the hair
dryer's pressure is higher.  The higher pressure means a larger upward force
on the balloon which is needed to exactly cancel the larger downward force
of gravity from the increased mass.
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(d) Finally, I will turn the hair dryer back to its "low" setting. What changes now? Why?

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Again the balloon floats down toward the end of the hair dryer to try to increase
the effective pressure it feels now that the hair dryer is blowing a lower
air pressure.  If the pressure from the end of the hair dryer is too low to
keep the balloon afloat, the balloon will sink all the way down and rest on
the hair dryer.
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(e) This has been a decent analogy for hydrostatic equilibrium in stars. Let's make the analogy a bit more explicit now. What do the two different settings on the hair dryer represent? What does the Earth's gravity represent? What do the two balloons represent?

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The two different settings represent two different values for the central pressure
of a star.  The Earth's gravity represents the internal gravity of a star.
The balloons represent two parcels of gas with different masses.  In the last
case, if the balloon sinks all the way down to the hair dryer, it is analogous
to a star running out of pressure support and collapsing (which is how stars
evolve and eventually explode)!
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3. **One Ring To Rule Them All**

For our next demo, we are going to observe the effect known as "limb brightening". In lecture we discussed the opposite effect, "limb darkening", quite a bit (see Figure 1).

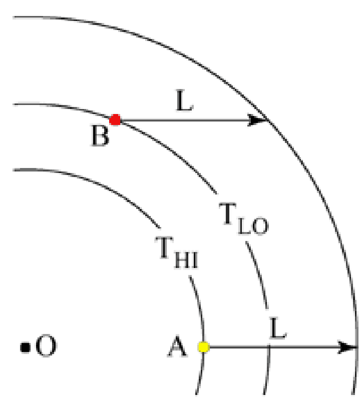

Figure 1: Limb Darkening in the Sun: When looking at the Sun in visible light, $\tau = 1$ is reached after you look a distance L through the Sun's atmosphere. When you look toward the center of the solar disk, this distance corresponds to point A. When you look at the edge (or "limb") of the Sun, this distance corresponds to point B, which you can see is at a larger distance from the center of the Sun (point O) than point A. The Sun's temperature decreases with radius in this region and thus the temperature at point A (T_{HI}) is greater than the temperature at point B (T_{LO}). Since $F \propto T^4$, the higher temperature areas will appear brighter than the lower temperature areas and so the limb of the Sun looks darker than the center of the solar disk.

Again using optical depth arguments, we will explain the phenomenon of "limb brightening". Let's turn off the lights...

(a) Here we have a (collapsed) Hoberman Sphere, to which we have attached white lights. The lights are roughly uniformly distributed on the surface of the sphere. Close one eye to simulate the lack of depth perception that astronomers have when observing objects on the celestial sphere. Can you see through the sphere? What do astronomers call such an object (in terms of its optical depth)?

`You cannot see through the sphere and thus we call it optically thick.`

(b) Does any part of the sphere look any brighter than any other part? Or, what shape do you see?

`The brightness should look quite uniform over the entire sphere and in fact, since we have no depth perception, it appears as a uniformly bright circle or disk.`

(c) Why don't we see limb darkening here (like we would for the Sun which is certainly also optically thick)?

`The light is all coming from the surface of the sphere and thus` $\tau = 1$ `at the surface of the sphere, all over the sphere. There is no temperature gradient or any difference at all between light coming from the limb or the center of the disk.`

(d) Now I will expand the sphere. Can you see through the sphere? What do astronomers call such an object (in terms of its optical depth)?

`You can now see through the sphere and thus we call it optically thin.`

(e) Does any part of the sphere look any brighter than any other part? Or, what shape do you see?

`The edges, or ''limb'', of the sphere should look brighter and it should appear as a bright ring with a little bit of light filling in the ring.`

(f) Why do we see limb brightening here?

`When we look along the limb of the sphere, we see a higher number of lights because of the projection effect of the edges of the sphere and thus the lights appear to have a higher spatial density at the limb. That's why we observe more flux at the limb as compared to looking straight at the center of the sphere.`

(g) If I rotate the sphere (allowing you to observe it from different viewing angles), do your observations change?

`The observations don't change as the sphere is rotated, since the lights are evenly spaced on the entire sphere, there is a spherical symmetry here and thus the sphere looks like a ring from any viewing angle.`

Lots of objects in astronomy are spherical and optically thin and thus appear as rings. One common example are planetary nebula (which we'll learn more about later this semester). Basically they are evolved low-mass stars that eject (or "burp" off) their outer layers of gas. There are some pictures of these objects around the room, including one of the most famous and easy to find in a small telescope, the Ring Nebula.

Also, expanding supernovae ejecta (the explosive deaths of stars) are usually fairly spherical and become optically thin very soon after the explosion. Probably the best-studied supernova of all time, SN 1987A, appears quite ring-like.

(h) If many astronomical objects appear to us as rings. How do we know that many of them are in fact optically thin spheres? (HINT: Both planetary nebulae and supernova ejecta are *expanding* optically thin spheres.)

If something appears as a ring, astronomers immediately want to know the overall 3D shape of the object. The third dimension that we miss on the sky is depth or distance along our line of sight. If an object is expanding (such as planetary nebulae or supernova ejecta) then we can take a spectrum of the object and we should see a blueshift from the material coming toward us (in the center of the ring). If the object is optically thin, then we should also see a redshift of the material going away from us (again in the center of the ring). As for the material in the ring itself, if it really is part of an expanding sphere, those would be the edges of the sphere expanding exactly perpendicular to our line of sight and thus we'd see neither a redshift nor a blueshift from that material. If we put all this together, we can see how the entire object is moving in space with respect to our line of sight and we can use this to get a handle on its 3D shape.