Ay 7A – Fall 2009
Section Worksheet 6
The Tao of τ[1]

We're now starting to move from somewhat familiar material (i.e., mechanics, the wave nature of light, mostly very physics-y type stuff) into much more unfamiliar, astronomy-type material. This theme will continue for much of the rest of the semester.
If you haven't done the readings lately, you really should...

1. **Some Tips on τ**

   Here are some extremely important (and basic) things to remember about astronomical radiation that we've discussed and that we'll discuss for most of the rest of the class:

   (a) The radiative transfer equation is

   $$\frac{dF}{dx} = -n\sigma F$$

   where F is the flux, x is the distance through a column of material, n is the number density of absorbers/scatterers, and σ is the cross-section of the material in the column to absorbing/scattering a photon.

   (b) Integrating the radiative transfer equation (and being careful about your limits of integration) gives

   $$F(x) = F_0 e^{-n\sigma x}$$

   where F_0 is the initial flux at the beginning of the column (i.e., $F_0 \equiv F(x = 0)$).

   (c) The mean free path is

   $$\lambda_{\text{mfp}} \equiv \frac{1}{n\sigma}$$

   This distance represents the average distance that a photon travels before being absorbed/scattered in a column of material with n number density of particles with σ cross-section to absorb/scatter photons.

   (d) The optical depth is

   $$\tau \equiv n\sigma x$$

   which makes the radiative transfer equations above nice and clean. Also notice that τ is unit-less.

   (e) $\tau = 1$ is special. When $\tau = 1$, $n\sigma x = 1$, and thus

   $$x_{\tau=1} = \frac{1}{n\sigma} = \lambda_{\text{mfp}}$$

   Since the distance into a column of material where $\tau = 1$ is equal to the mean free path of the photon in the column, this is the point in the column where photons have pretty good chances of **both** getting scattered/absorbed AND passing through unimpeded. Therefore, $\tau = 1$ defines a sort of boundary between photons "free-streaming" through the column and photons getting stuck (i.e., scattered or absorbed) in the column.

[1] Title courtesy of former Ay 7A GSIs John Johnson and Renbin Yan.

(f) We call $\tau \gg 1$ "optically thick". This is because when τ is large, the exponential factor in the radiative transfer equation is going to be very small, so the final flux will be much smaller than its initial value. Another way to think about this is that $x \gg \lambda_{\text{mfp}}$, so many of the photons will be scattered/absorbed by the time they get a distance x through the column. Also, when an object is optically thick, astronomers often say that we "see to $\tau = 1$" since the photons we observe from an optically thick object are escaping that object (and traveling to our eye or detector) from where $\tau \approx 1$ in the object.

(g) We call $\tau \ll 1$ "optically thin". This is because when τ is small, the exponential factor in the radiative transfer equation is going to be very close to 1, so the final flux will be very close to its initial value. Another way to think about this is that $x \ll \lambda_{\text{mfp}}$, so many of the photons will **NOT** be scattered/absorbed by the time they get a distance x through the column.

2. **Fog, Fog, Go Away...**

In this question we'll explore why astronomers use the dimensionless τ instead of more physical values like distance (i.e., x). Hopefully this will make you appreciate τ.

(a) Let's think a bit more about what it means to have an optical depth of 1 (i.e., $\tau = 1$). If there is some amount of flux incident on a cloud of $\tau = 1$, what percentage of it makes it to you, the observer? In general, we say that you can see out to an optical depth of 1, because beyond that things are too opaque (i.e., too optically thick). Hopefully this makes some sense given what you just calculated.

`Using the radiative transfer equation, when` $\tau = 1$, $F(\tau = 1) = F_0 e^{-1} \approx 0.37 F_0$. `This means that about 37% of the original flux makes it through the cloud.`

(b) Consider a **really** foggy day in Berkeley (which happens fairly regularly). Estimate a distance that corresponds to $\tau = 1$. Think about wandering around campus in the fog.

`On a really foggy day in Berkeley, the farthest you can see is maybe several meters. Beyond that, the fog is too dense for you to see anything. Closer than that you can see pretty well. Thus, to an OK approximation we can say that about 10 meters is the distance that corresponds to` $\tau = 1$.

(c) Now consider a slightly hazy day, such that you can barely see the Golden Gate bridge from campus. Estimate the distance that corresponds to $\tau = 1$.

`If you can just make out the Golden Gate Bridge, then the distance to the Bridge is the distance corresponding to` $\tau = 1$. `I don't know the actual number, but to a factor of a few, 10 kilometers seems reasonable.`

(d) Now suppose in both cases that the obscuration is due to water droplets in the air. What is the ratio of the number density of water molecules in the air in the above cases?

`In both cases` $\tau_F = \tau_H = 1$, `where the` F `subscript is for the foggy day and the` H `subscript is for the hazy day. Thus:`

$$n_F \sigma_{\text{water}} d_F = n_H \sigma_{\text{water}} d_H$$

`and in the equation above` σ_{water} `cancels out since water droplets are responsible for the obscuration in both situations. Thus,`

$$\frac{n_F}{n_H} = \frac{d_H}{d_F} = \frac{10 \text{ km}}{10 \text{ m}} = 1000$$

`So, on a really foggy day, the density of water droplets in the air is something like 1000 times larger than it is on a hazy day.`

3. **Cloud Fluxes: Size Matters...Sometimes**

Suppose we have an interstellar cloud of gas shaped like a cylinder and we observe it end-on (so that we see a face-on circle of area A) and let's call the length of the cloud L_1. It is at a temperature T, has a total optical depth τ_1, and has number density n and the particles' cross-section is σ. Call this cloud Cloud 1. Now let's say we have a similar cloud called Cloud 2 but its length is $2L_1$ (but it still has T, n, and σ).

   (a) If $\tau_1 \gg 1$, how do the observed fluxes of the two clouds compare?

   `Note that` $\tau_2 = 2\tau_1$ `since the only physical difference between the two clouds is their length (i.e.,` x `in our definition of` τ`). In this case, both clouds are optically thick -- in either cloud, photons reach us from only a shallow depth into the cloud (corresponding to the place where` $\tau(z) \approx 1$`). Thus it makes no difference whether the cloud's linear length is` L_0 `or` $2L_0$ `-- no photons will reach us from anywhere near that deep into the cloud. The emergent fluxes will thus be about equal (since their temperatures are equal):` $F_1 \approx F_2$

   (b) What about if $\tau_1 \ll 1$?

   `In this case, both clouds are optically thin -- in either cloud, there is very little absorption taking place and we ''see right through'' the entire cloud. Therefore we can neglect absorption and consider only the emission. The amount of emission in the cloud is proportional to the number of particles doing the emitting, so the cloud that's twice as long (along our line of sight) will have twice the flux:` $2F_1 \approx F_2$

4. **The Universe is Dusty**

Now we will consider a cloud like the ones above, but instead of being made of mostly hydrogen gas we will make it mostly consist of dust (i.e., grains of rock or ice about the size of the particles in cigarette smoke – about 1μm in radius). The dust will not emit spectral lines like hydrogen gas (mainly due to the fact that the electrons that cause the hydrogen lines are all wrapped up in chemical bonds in the dust particles). However, the dust does interact with starlight that shines on these clouds.

We used the "boat on the ocean" analogy in class when discussing Rayleigh scattering. The short version is that if a small wavelength wave is traveling on the ocean and hits a boat, it will get messed up (i.e., scattered). If a huge wavelength tsunami wave is traveling on the ocean and encounters a boat, it will pass right on through like the boat wasn't even there (and the boat will just bob up and down as the wave passes by). Now this isn't an exact analogy, but it helps people remember this basic physics result (referred to as Rayleigh scattering): **shorter wavelength waves scatter more than longer wavelength waves, where "shorter" and "longer" are with respect to the size of the obstacle that the wave is encountering**.

   (a) Using what you know about Rayleigh scattering, what color of the visible spectrum of light will be most bounced around (i.e., scattered) by a dust cloud?

   `Blue/indigo/violet/purple.`

   (b) What color of visible light will travel most unimpeded straight through a dust cloud (i.e., transmitted)?

   `Red.`

   (c) If we have a star sitting near a cloud of dust (but not directly behind it along our line sight), what color will the cloud appear to our eyes? This is called a "reflection nebula".

   `Blue -- the human eye isn't very sensitive to purple/violet light.`

(d) Use this result to explain why the clear, sunny sky is blue and the Sun at sunset is reddish. Feel free to draw a diagram.

`If you look at a random piece of the sky where the Sun isn't, then you are seeing light from the Sun that is being scattered by gas in Earth's atmosphere. This scattered light is going to be much more blue than red.`

`If you look at the Sun during a sunset or sunrise (which is fairly safe as long as there's a fair bit of haze or clouds near the horizon AND you don't stare for too long), then it appears reddish (certainly more red than it does when it's higher in the sky and appears more yellowish). This is because when the Sun is close to the horizon, the sunlight's path length is longer through the atmosphere than it is when it's high in the sky. Therefore there are more atmosphere particles between our eyes and the Sun when it's near the horizon and thus there are more chances for light to get scattered out of our line of sight to the Sun. If we increase the number of Rayleigh scatterings, we know that we are preferentially losing more blue light than red light. So if you take the yellow Sun and increase the overall amount of Rayleigh scattering, you get left with a reddish Sun!`

`Note that this argument (and Rayleigh scattering in general) only works because oxygen and nitrogen molecules in our atmosphere are comparably sized to the wavelengths of visible light.`

5. **Got Rayleigh?**

(a) We start with a fish tank filled with plain old water. What color is the light from the flashlight? `Yellow/white.`

(b) What color is the water due to the flashlight? `Clear-ish.`

(c) We now add a little milk to the tank and stir. What color is the light from the flashlight (i.e., when the flashlight is pointing along your line of sight)? `Reddish.`

(d) What color is the water due to the flashlight (when the flashlight is NOT pointing along your line of sight)? `Blueish.`

(e) Again using Rayleigh scattering, explain your observations.

`This is nearly a perfect analogy to the Sun and sky in the previous problem. Here, the flashlight is acting like the Sun and the milk particles (which are big, fatty organic molecules) are acting like the atmosphere.`

(f) We now add quite a bit of milk. What color is the flashlight's light? `Brownish/yellowy/milky/gray.`

(g) What color is the water? `Brownish/yellowy/milky/gray.`

(h) We've clearly been increasing the optical depth of the tank since the flashlight appears dimmer than when we started. What physical parameter have we been changing that affects the optical depth?

`We've been increasing the number density,` n`, by adding more milk molecules.`

(i) Does the scattering depend on wavelength anymore?

`No. When we have this many milk particles in the tank, all of the visible light is being scattered basically the same amount (i.e., a lot), no matter what the color/wavelength. This type of scattering is seen (and assumed) in many astrophysical sources and astronomers refer to it as ''gray'' scattering. Do you see why?!?!`