Ay 7A – Fall 2009
Section Worksheet 5
The Serious Science of Studying Solar Spectra[1]

In this worksheet we are going to find the relative abundance of calcium in the Sun's atmosphere by looking at the solar spectrum. This is a very important technique to understand because it gives us most of the information we know about the composition of stars and other astronomical objects.
If you look at the "strongest" absorption lines in the spectrum of the Sun, one thing you may notice is that the CaII H & K lines are stronger than the Hα line ("stronger" in this context means a deeper and wider absorption line). Check out Figure 1 to see these lines and notice that they are relatively close in wavelength (i.e. energy). A typical assumption here might be that there is more calcium than hydrogen in the Sun. However, I expect all of you already know that there is much more hydrogen than calcium in the Sun. If you didn't know that, you should after doing this worksheet! Now we're going to explore some of the factors that enter into how strong a spectral line is.

1. Calculate the energy in eV of the ground state and first two excited states of hydrogen. Try to remember how to do this without looking up the equation. (Hint: the ionization energy of hydrogen is 13.6 eV, a number which you might want to memorize if you are going into astronomy.) The Hα line is the transition from the n=2 to the n=3. What is ΔE for this transition?

   `The energy of levels of the hydrogen atom are given by` $E_n = (-13.6 \text{ eV})/n^2$. `This is something that would be good for you all to remember. The ground state energy is -13.6 eV. The first excited state energy is -3.4 eV. The second excited state energy is -1.5 eV. The transition between the first and second excited states has` $\Delta E = -1.5 - -3.4 = 1.9$ `eV.`

2. In order to find out the abundance of calcium relative to hydrogen we need to compare the strengths of their spectral lines. Let's start with Hα, and think about what determines the strength of this line. Assume that the hydrogen is essentially all neutral, i.e. not ionized, which in astronomy lingo we would write as HI. The effective temperature of the sun is 5770 K.

   (a) What is the average energy of an atom in the Sun in eV? Recall that $k = 1.38 \times 10^{-16}$ erg/K and 1 eV is about 1.60×10^{-12} erg.

   `The average energy of a particle in a gas that is in thermal equilibrium is approximately equal to` kT. `For the Sun,` $kT \approx 7.96 \times 10^{-13}$ `erg, and then converting to eV, we get that` $kT \approx 0.5$ `eV.`

   (b) The determining factor here is going to be the fraction of hydrogen atoms with an electron in the first excited state (since these are the only hydrogen atoms that can emit Hα light) to the total number of hydrogen atoms. Calculate this fraction using Boltzmann's equation:

   $$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-(E_2 - E_1)/kT}$$

   where N_1 is the number of (neutral) hydrogen atoms with its electron in the n=1 (ground) state and N_2 is the number of (neutral) hydrogen atoms with its electron in the n=2 (first excited) state. (Hint: at the effective temperature of the Sun, pretty much all of the hydrogen is either in the ground state or first excited state and recall that the degeneracy of levels in the hydrogen atom is given by $g_n = 2n^2$)

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[1] Try saying that five times fast!

We want to find N_2/N_{total}, but this is not exactly what the Boltzmann equation tells us. What it tells us is:

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-(E_2-E_1)/kT}$$

This means that we have to be slightly tricky and rearrange N_2/N_{total} into something we can find, like so:

$$\frac{N_2}{N_{\text{total}}} = \frac{N_2}{N_1+N_2} = \frac{N_2/N_1}{1+N_2/N_1}$$

Here we have made use of the fact that almost all hydrogen is either in the ground state or the first excited state at the temperature of the sun. $kT \approx 0.5$ eV (as we saw in part (a)) and the difference between the ground state and first excited state energies is 10.2 eV (from the previous question). Plugging these numbers and the degeneracy of the levels in, we find that $N_2/N_{\text{total}} \approx 5.5\times10^{-9}$. This means that one out of every $\sim 1.8\times10^{8}$ hydrogen atoms is in the first excited state.

(c) Explain the assumptions that went into getting this fraction!

The two major assumptions to think about here are that we aren't taking into account the ionization of hydrogen and that we are assuming that all hydrogen is either in the ground or first excited state. The second assumption is very good. As we've calculated with Boltzmann's equation very nearly all of the hydrogen is in the ground state, and the amount in states higher than the first excited state is going to be negligibly small. You can calculate this if you are interested. The assumption that the hydrogen is mostly neutral is also a good assumption. You can calculate using the Saha equation that, at the temperature of the Sun, $n_{II}/n_I \approx 7.5 \times 10^{-5}$. So both of these assumptions are OK.

3. Now let's talk about calcium. In the atmosphere of the Sun basically all of the calcium is CaII (i.e. singly-ionized). The ionization energy of calcium is 6.11 eV. Compare this energy to the hydrogen ionization energy and you'll see why the hydrogen is almost all neutral (HI) while the calciums is almost all singly-ionized (CaII). The CaII K line is the transition from the ground state to first excited state. The degeneracies of these levels are $g_1 = 2$ and $g_2 = 4$.

(a) The wavelength of the CaII K line is 3933 Å. What is the ΔE of this transition in eV?

The easiest way to do this is to do a ratio with the Hα line. The wavelength of Hα is 6563 Å, and the ΔE for Hα is 1.9 eV (as calculated in the first problem). Comparing this to the wavelength of 3933 Å for calcium, you find that $\Delta E = 3.17$ eV. You could also have used the standard $\Delta E = hc/\lambda$ here and gotten the same answer.

(b) To determine the strength of the line, we need to find the fraction of all CaII atoms which have an electron in the ground state, so go for it. Again, make some simple assumptions to simplify your answer.

Here we use Boltzmann's equation with the given degeneracies and the ΔE we just calculated. Plugging this information into the equation tells us that $N_2/N_1 = 3.53 \times 10^{-3}$ for CaII in the Sun. However, we are interested in the total fraction of all CaII atoms with an electron in the ground state:

$$\frac{N_1}{N_{\text{total}}} = \frac{N_1}{N_1+N_2} = \frac{1}{1+N_2/N_1}$$

Plugging in the value we calculated for N_2/N_1 we find that $N_1/N_{\text{total}} \approx 0.996$ indicating that almost all CaII ions are in the ground state and therefore available for the H & K transitions.

Just to remind you, here we have made two assumptions. One, that calcium is almost all singly-ionized. Two, that the singly ionized calcium has almost all electrons in either the ground or first excited states. You can check both of these assumptions, should you so desire, using the Boltzmann and Saha equations.

4. Now let's assume that the "strength" of a spectral line only depends on the fraction of atoms with electrons in the appropriate energy level, as well as the abundance of the atom creating the line. This isn't quite true, but the details are unimportant for us right now. Given the solar spectrum in Figure 1, assume that the calcium line is 400 times stronger than the hydrogen line. Using the fractions you calculated above, what are the relative abundances of the two elements?

   So basically we want to compare the strengths of the two lines. Let's make a ratio of the strengths! Calcium on top and Hα on the bottom.

$$\frac{S_{\text{CaII}}}{S_{\text{H}\alpha}} = 400 = \frac{f_{\text{Ca}}}{f_{\text{H}}} \times A = \frac{0.996}{5.5 \times 10^{-9}} \times A$$

   where the f's are the fraction of a particular atom that can participate in the transition, and $A = N_{\text{Ca}}/N_{\text{H}}$ is the relative abundance of the two atoms (what we're solving for). If you plug in the numbers you've already calculated, you'll find that $A \approx 2.2 \times 10^{-6}$. This means that for every 1 calcium atom, there are about 450,000 hydrogen atoms in the Sun!

5. In general, taking a spectrum of the entire blackbody curve of a star (i.e. the spectrum at *all* wavelengths) is very impractical. Usually, stellar spectroscopy involves a very small region of the spectrum, from which you can't usually determine the peak of the blackbody curve to find the star's temperature. Given a relatively small region of the spectrum, what is one way to determine both the temperature and abundance of an element simultaneously?

   The key to doing this is to examine more than one spectral line for a given species of atom. If you can measure the strength of two different absorption features of the same species of atom, then you have 2 equations for 2 unknowns (the temperature and the abundance) and you can solve for both.

   For example, you can repeat exactly what we did above (for the CaII K line), without plugging in an actual value for the Sun's temperature. Thus you would have the equation

$$\frac{S_{\text{CaII K}}}{S_{\text{H}\alpha}} = \frac{f_{\text{CaII K}}}{f_{\text{H}}} \times A$$

   and the only unknowns would be temperature T (which appears in the Boltzmann Equation and thus appears in both $f_{\text{CaII K}}$ and f_{H}) and relative abundance, A. Then you can repeat this process for a different CaII line (perhaps the transition from the ground state, n=1, to *second* excited state, n=3). As long as you know the wavelength (or energy) of that transition and the degeneracy of the second excited state, you can exactly repeat our above calculation and get the equation:

$$\frac{S_{\text{CaII } 1\rightarrow 3}}{S_{\text{H}\alpha}} = \frac{f_{\text{CaII } 1\rightarrow 3}}{f_{\text{H}}} \times A$$

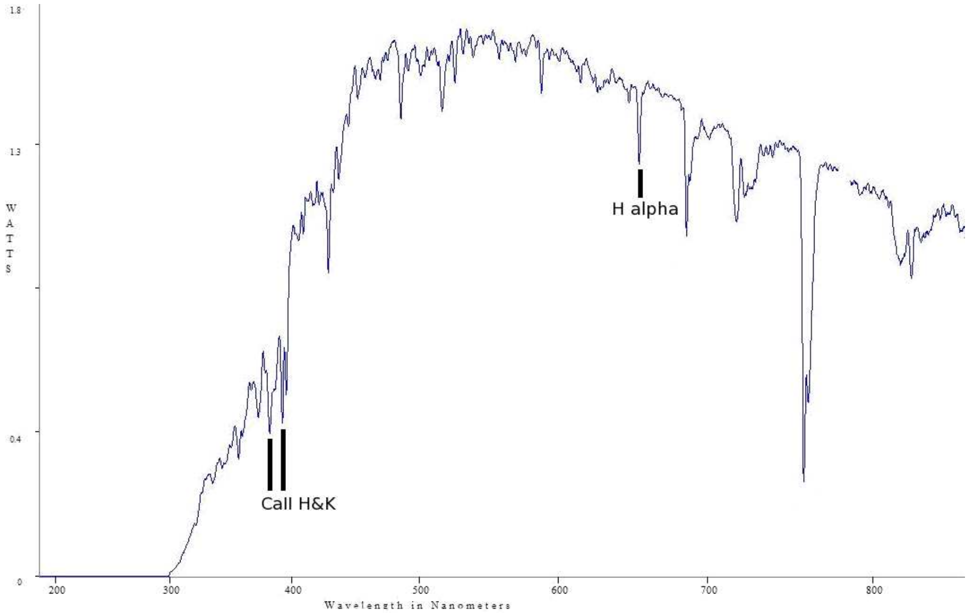

Figure 1: The region of the solar spectrum showing the CaII H & K lines and the Hα line.

and again the only unknowns would be temperature T (which appears in both $f_{\mathrm{CaII}\ 1\rightarrow 3}$ and f_{H}) and relative abundance, A.

Now you have two equations with two unknowns and you can solve for both simultaneously! In reality this is more complicated (isn't it always?), and what is generally done is a fit to many spectral features of many different atoms simultaneously.